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SHORTER COMMUNICATIONS

AN ANALYTICAL STUDY OF NATURAL CONVECTION IN A VERTICAL OPEN TUBE

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NOMENCLATURE

Gr, Grashof number;
J_0, J_1, J_2, Bessel functions of the first kind of order zero, one and two;
p, pressure;
Pr, Prandtl number;
Q, heat dissipation rate;
T, temperature;
u, v, fluid velocities in the flow and radial directions;
x, r, flow and radial directions.

Greek symbols

α_n, characteristic values defined by $J_2(\alpha_n) = 0, \alpha_n \neq 0$;
β_n, characteristic values defined by $J_0(\beta_n) = 0$;
ξ, independent variable, $x/u_i\,Gr$.

Subscript

i, tube inlet.

THE PURPOSE of this communication is to analyze the development of free convection in finite vertical tubes by an analytical method which is based upon a slug-flow linearization of the governing boundary-layer type equations. The resulting equations are solved by means of Laplace transforms to give simple closed form expressions for the flow variables. The present results are also compared with available numerical results found in the literature [1, 2].

BASIC EQUATIONS

The following basic assumptions are made to obtain the combined velocity and temperature fields and the heat-transfer characteristics of flow in a vertical circular tube maintained at a constant temperature higher than the ambient temperature: (1) The fluid enters the tube with ambient temperature and uniform velocity profile. (2) The viscous dissipation terms can be neglected. (3) The density varies only in the gravity force term (the Boussinesq assumption). (4) All other physical properties of the fluid are constant.

Applying the usual boundary-layer assumption to the governing differential equations (continuity, momentum and energy) yields the following steady, incompressible, two-dimensional boundary-layer equations:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{1}{Gr}\left(T + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r}\right) - \frac{\mathrm{d}p}{\mathrm{d}x}, \tag{2}$$

and

$$u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial r} = \frac{1}{Gr\,Pr}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right). \tag{3}$$

The wall boundary conditions are taken as zero velocity and prescribed temperature. At the center of the tube the usual symmetry conditions prevail. At the inlet of the tube the velocity and temperature are taken as constants. The initial velocity is not an independent parameter of the flow problem. If this initial velocity is termed u_i then the initial pressure for natural convection flow is given by application of Bernoulli's equation as $p_i = -u_i^2/2$ [3, 4]. Finally, setting the pressure equal to zero at the tube exit gives the unknown initial velocity u_i. This boundary condition also makes the streamlines in the emerging flow parallel.

An additional integral continuity equation is required because the r-momentum equation has been eliminated through the boundary layer assumptions, and it is given by

$$2\int_0^1 ur\,\mathrm{d}r = u_i. \tag{4}$$

LINEARIZED SOLUTION OF EQUATIONS

The governing equations (1)–(4) can not be solved analytically for pre-developed flow. Although a numerical solution is possible [1, 2], to arrive at a tractable problem by analytical methods the momentum and energy convective terms will be linearized following analysis by Sparrow *et al.* [5] for the isothermal-entrance-region problem. The method is essentially a slug-flow linearization of the governing equations, i.e.

$$u\frac{\partial}{\partial x} + v\frac{\partial}{\partial r} \equiv u_i\frac{\partial}{\partial x}. \tag{5}$$

With the introduction of a new independent variable $\xi = x/(u_i\,Gr)$ the linearized equations become

$$\frac{\partial u}{\partial \xi} = T + \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} - \frac{1}{u_i}\frac{\mathrm{d}p}{\mathrm{d}\xi}, \tag{6}$$

$$\frac{\partial T}{\partial \xi} = \frac{1}{Pr}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r}\right), \tag{7}$$

and

$$2\int_0^1 ur\,\mathrm{d}r = u_i. \tag{8}$$

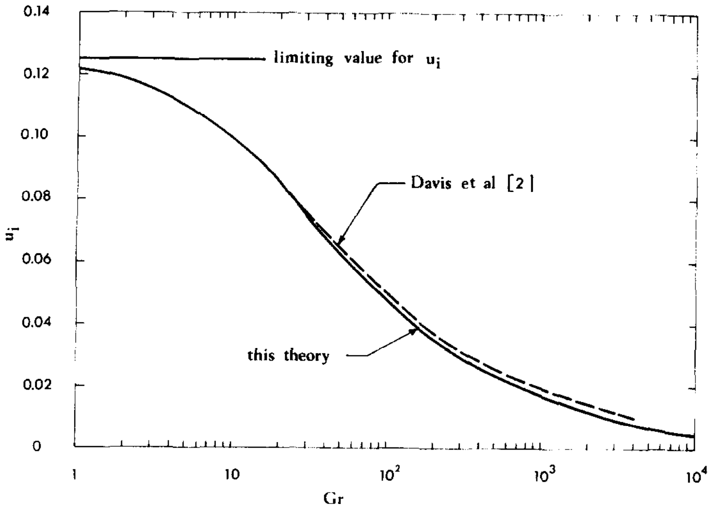

FIG. 1. Variation of inlet velocity u_i with Gr for $Pr = 0.7$.

The boundary conditions may now be written as:

$$r = 0: \quad \frac{\partial u}{\partial r} = 0, \quad \frac{\partial T}{\partial r} = 0, \tag{9}$$

$$r = 1: \quad u = 0, \quad T = 1, \tag{10}$$

$$\xi = 0: \quad u = u_i, \quad T = 0, \quad p = -\frac{u_i^2}{2}, \tag{11}$$

$$\xi = \frac{1}{u_i\,Gr}: \quad p = 0. \tag{12}$$

Equations (6)–(11) are now solved by a Laplace transform technique to yield simple closed form expressions for the flow variables T, u and p in terms of the initial velocity u_i, i.e.

$$T(\xi,r) = 1-2\sum_{n=1}^{\infty}\frac{J_0(\beta_n r)\,e^{-\frac{\beta_n^2}{Pr}\xi}}{\beta_n J_1(\beta_n)}, \tag{13}$$

$$\begin{aligned} u(\xi,r) = 2u_i\Bigg\{&1-r^2-2\sum_{n=1}^{\infty}\frac{e^{-\alpha_n^2\xi}}{\alpha_n^2}\left[1-\frac{J_0(\alpha_n r)}{J_0(\alpha_n)}\right] \\ &+\frac{2}{(Pr-1)}\Bigg\{2\sum_{n=1}^{\infty}\frac{J_2[(Pr)^{1/2}\cdot\alpha_n]\,e^{-\alpha_n^2\xi}}{\alpha_n^4 J_0[(Pr)^{1/2}\cdot\alpha_n]}\left[1-\frac{J_0(\alpha_n r)}{J_0(\alpha_n)}\right] \\ &-2Pr\sum_{n=1}^{\infty}\frac{J_0\left[\frac{\beta_n}{(Pr)^{1/2}}\right]e^{-\frac{\beta_n^2}{Pr}\xi}}{\beta_n^4 J_2\left[\frac{\beta_n}{(Pr)^{1/2}}\right]}\left[1-\frac{J_0\left(\frac{\beta_n}{(Pr)^{1/2}}r\right)}{J_0\left(\frac{\beta_n}{(Pr)^{1/2}}\right)}\right] \\ &-Pr\sum_{n=1}^{\infty}\frac{J_0(\beta_n r)\,e^{-\frac{\beta_n^2}{Pr}\xi}}{\beta_n^3 J_1(\beta_n)}\Bigg\}, \end{aligned} \tag{14}$$

and

$$\begin{aligned} p(\xi) = u_i^2\left[-8\xi-\frac{5}{6}+4\sum_{n=1}^{\infty}\frac{e^{-\alpha_n^2\xi}}{\alpha_n^2}\right] &+\frac{4u_i}{(Pr-1)}\Bigg\{\frac{(Pr-1)}{4}\left(\xi-\frac{Pr}{6}\right) \\ &-\sum_{n=1}^{\infty}\frac{J_2[(Pr)^{1/2}\alpha_n]\,e^{-\alpha_n^2\xi}}{\alpha_n^4 J_0[(Pr)^{1/2}\cdot\chi_n]}+Pr\sum_{n=1}^{\infty}\frac{J_0\left[\frac{\beta_n}{(Pr)^{1/2}}\right]e^{-\frac{\beta_n^2}{Pr}\xi}}{\beta_n^4 J_2\left[\frac{\beta_n}{(Pr)^{1/2}}\right]}\Bigg\} \end{aligned} \tag{15}$$

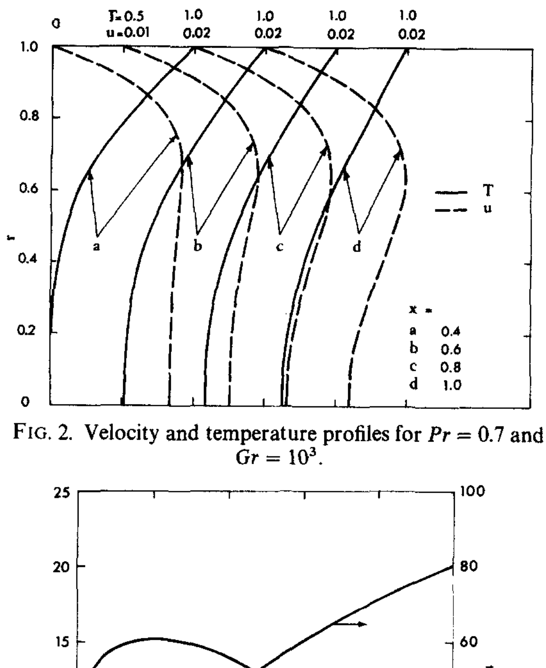

FIG. 2. Velocity and temperature profiles for $Pr = 0.7$ and $Gr = 10^3$.

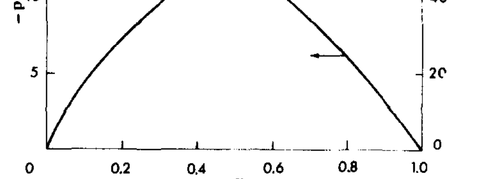

FIG. 3. Fluid pressure and heat absorbed Q vs tube length for $Pr = 0.7$ and $Gr = 10^3$.

where the characteristic values α_n are found from $J_2(\alpha_n) = 0$, $\alpha_n \neq 0$ and the characteristic values β_n from $J_0(\beta_n) = 0$.

In order to complete the solution the parameter u_i must be determined as a function of independent parameters, Pr and Gr. This is accomplished by first getting an implicit expression for u_i from equation (15) by use of equation (12). Solving the resulting algebraic equation iteratively for u_i and then substituting into equations (13) through (15) permits explicit solutions for T, u and p.

The heat absorbed by the fluid rising in the tube can be written as

$$Q = 2\int_0^1 uTr\,dr. \quad (16)$$

RESULTS AND DISCUSSION

In order to justify the validity of the present linearized method, the results were compared with available exact numerical solutions to the problem. Figure 1 shows the calculated inlet velocity u_i vs Gr for $Pr = 0.7$ as well as the values given by Davis and Perona [2]. The numerical results of Kageyama *et al.* [1] for $Pr = 0.72$ is very much similar to that of [2], and are not shown in the figure. The numerical solutions correspond to zero inlet pressure, while the actual pressure must be lower since the fluid has been accelerated from rest [3, 4]. The good agreement shown between the results of the linearized version of the boundary layer equations with the exact numerical solutions verifies the propriety of the present method of analysis.

Representative temperature and velocity profiles at four levels of x are shown in Fig. 2, while the heat flux and pressure levels as a function of axial position are shown in Fig. 3. The pressure defect is decreased by large buoyancy forces for large x.

It must be noted, however, that the use of boundary layer type of equations in the analysis is limited to cases where Gr is not too large. Otherwise, the velocity profiles will indicate a downward flow at the center of the tube exit, and boundary-layer-type equations used would not be valid for this flow behavior. Finally, the Oseen type of linearization of the governing equations should only relate to physical cases where the length to diameter ratios of tubes considered are moderately large.

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CONVECTIVE HEAT TRANSFER TO LAMINAR FLOW OVER A PLATE OF FINITE THICKNESS

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NOMENCLATURE

b,	plate thickness;
Br_x,	local Brun number defined by equation (1);
K,	thermal conductivity;
Pr,	Prandtl number;
q,	local heat flux;
Q,	transformed heat flux defined by equation (14);
Re_x,	local Reynolds number;
T,	temperature;
u, v,	longitudinal and transverse components of velocity in boundary layer;
x, y,	Cartesian coordinates;
X,	transformed coordinate defined by equation (18);
Z	transformed coordinate defined by equation (10);

Greek symbols

α,	thermal diffusivity;
β,	parameter defined by equation (17);
δ,	boundary-layer thickness;
ξ,	transformed distance defined by equation (9);
τ,	shear stress;
μ,	absolute viscosity;
ν,	kinematic viscosity;
θ,	dimensionless temperature defined by equation (6);
Φ,	dimensionless temperature, $\Phi = 1-\theta$.

Subscripts

ω,	refers to wall surface in contact with fluid;
b,	refers to wall surface at constant temperature;
f,	refers to fluid;
s,	refers to solid;
T,	refers to thermal boundary layer;
x,	refers to local values;
∞,	refers to mainstream flow;
0,	refers to values of Nu_x at $Br_x = 0$.

INTRODUCTION

IN THE usual formulation of the problem of heat transfer to flow over a flat plate, boundary conditions are specified at the upper surface of the plate which is in contact with the fluid. If, however, the boundary conditions are specified over the lower surface of the plate, the effect of plate resistance, if signigicant, must be included in the analysis resulting in a conjugate heat-transfer problem. This represents a more realistic approach and analyses of this type have recently received increased attention resulting in publication of a number of papers [1]. A formulation of such problems was originally presented by Luikov [2] and analytical methods of solution of certain conjugate problems were given by Luikov, Aleksashenko and Aleksashenko [3]. More recently Luikov [4] presented a solution of the problem of heat transfer to laminar flow over a plate of finite thickness with the lower surface of the plate maintained at a uniform constant